Certain Semigroups on Banach Function Spaces and Their Adjoints

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In this note C_0-semigroups on Banach function spaces are studied. In the first part we are concerned with the problem under what conditions the semigroup dual space is a subspace of the associate space. In the second part we investigate when a multiplication operator of the form $A_h f = hf$ generates a C_0-semigroup. For those h for which this is the case we give a representation for the semigroup dual space.

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1. Preliminaries

Let (Ω, Σ, μ) be a σ-finite measure space and let $L^0(\mu)$ denote the linear space of μ-measurable functions on Ω which are finite a.e. As usual μ-a.e. equal functions are identified. A linear subspace X of $L^0(\mu)$, equipped with a norm $\|\cdot\|$, is called a *Banach function space* (over (Ω, Σ, μ)) if X is a Banach space with respect to $\|\cdot\|$ and $f \in L^0(\mu)$, $g \in X$ with $|f| \leq |g|$ a.e. implies that $f \in X$ and $\|f\| \leq \|g\|$. Note that every Banach function space is a Banach lattice. For the basic theory concerning Banach function spaces we refer to the books [3], [8], [9]. We will recall some of the relevant facts.

We say that X *is carried by* Ω if there is no subset E of Ω of positive measure with the property that $f = 0$ a.e. on E for all $f \in X$, or equivalently if for every $E \subset \Omega$ of positive measure there is a subset $F \subset E$ of positive measure such that the characteristic function χ_F belongs to X. Ω always contains a subset Ω_0 such that X is carried by $\Omega \backslash \Omega_0$. Therefore we will assume henceforth without loss of generality that X is carried by Ω.

The *associate space* (sometimes called the Köthe dual) of X is defined by

$$X' = \{g \in L^0(\mu) : \int_\Omega |fg| \, d\mu < \infty, \forall f \in X\}.$$

X' is a Banach function space with respect to the norm given by

$$\|g\| = \sup_{\|f\| \leq 1} \left| \int_\Omega fg \, d\mu \right|.$$

Every $g \in X'$ defines a bounded linear functional $\phi_g \in X^*$ via the formula

$$\langle \phi_g, f \rangle = \int_\Omega fg \, d\mu, \qquad \forall f \in X.$$

We have $\|g\|_{X'} = \|\phi_g\|_{X^*}$. Therefore X' can be identified with a closed subspace of X^*. In fact X' is even a band in X^*.

The norm of X is called *order continuous* if $f_n \downarrow 0$ in X implies $\|f_n\| \downarrow 0$. X has order continuous norm if and only if $X' = X^*$.

A linear functional $\phi \in X^*$ is called *order continuous* if $f_n \downarrow 0$ in X implies $\langle \phi, f_n \rangle \to 0$. One can show that $\phi \in X^*$ is order continuous if and only if $\phi \in X'$. Finally, a positive linear operator $T : X \to X$ is called *order continuous* if $f_n \downarrow 0$ implies $Tf_n \downarrow 0$.

We will also need some terminology on adjoint semigroups. See [1], [5], [6] for more details. Let $T(t)$ be a C_0-semigroup of operators on a Banach space X. The *adjoint* semigroup on X^* is defined by $T^*(t) = (T(t))^*$. $T^*(t)$ need not be strongly continuous. We define

$$X^\odot = \{x^* \in X^* : \lim_{t \downarrow 0} \|T^*(t)x^* - x^*\| = 0\}.$$

$X^\odot$ is a norm-closed, weak*-dense subspace of X^*. In fact, if A is the generator of $T(t)$, then $X^\odot$ is precisely the norm-closure of $D(A^*)$. $X^\odot$ is invariant under $T^*(t)$, so the restrictions $T^\odot(t)$ of $T(t)$ to $X^\odot$ define a C_0-semigroup on $X^\odot$. Applying the same construction to this semigroup, we define $X^{\odot\odot} = (X^\odot)^\odot$. The map $j : X \to X^{\odot *}$,

$$\langle jx, x^\odot \rangle := \langle x^\odot, x \rangle$$

is actually an embedding which maps X into $X^{\odot\odot}$. In case $jX = X^{\odot\odot}$ we say that X is *sun-reflexive with respect to* $T(t)$. It is well-known that this is the case if and only if the resolvent $R(\lambda, A)$ is weakly compact.

If $T(t)$ is a C_0-semigroup on a Banach function space X, then one may ask under what conditions we have $X^\odot \subset X'$. Trivially, this is true when X has order continuous norm. Recall that a Banach lattice is said to be *σ-Dedekind complete* if every countable subset that is bounded from above has a supremum. Every Banach function space is σ-Dedekind complete.

Lemma 1.1. *Suppose $T(t)$ is a C_0-semigroup on a Banach function space X. Then the band generated by $X^\odot$ is equal to X^*.*

Proof: By a result of Schaefer [7] a band in the dual of a σ-Dedekind complete Banach lattice is sequentially weak*-closed. Let Y denote the band in X^* generated by $X^\odot$ and take $\phi \in X^*$ arbitrary. Since

$$\lambda_n R(\lambda_n, A)^* \phi \to \phi \qquad \text{weak}^*$$

for any sequence $\lambda_n \to \infty$ in $\varrho(A)$, and since $\lambda_n R(\lambda_n, A)^* \phi \in X^\odot$, it follows that $\phi \in Y$ and hence $Y = X^*$. ////

Theorem 1.2. *Suppose X is a C_0-semigroup on a Banach function space X. Then $X^\odot \subset X'$ if and only if X has order continuous norm.*

Proof: If X has order continuous norm, then $X' = X^*$, so trivially $X^\odot \subset X'$ holds. Conversely, suppose $X^\odot \subset X'$. Since X' is a band in X^*, by Lemma 1.1 we have $X^* \subset X'$, forcing $X' = X^*$. ////

We remark that the same result holds mutatis mutandis for any σ-Dedekind complete Banach lattice. The equivalent hypotheses of Theorem 1.2 are always fulfilled in the sun-reflexive case. This is the content of Theorem 1.4 below.

Recall that a Banach space is called *weakly compactly generated (WCG)* if it is the closed linear span of one of its weakly compact subsets.

Lemma 1.3. *Suppose a Banach space X is sun-reflexive with respect to a C_0-semigroup. Then X does not contain a subspace isomorphic to l^∞.*

Proof: Suppose the contrary and let Y be a subspace of X which is isomorphic to l^∞. Since l^∞ is complemented in every Banach space containing it as a subpace [4, Prop. I.2.f.2], it follows that Y is complemented in X. Since the resolvent $R(\lambda, A)$ is weakly compact and $R(\lambda, A)(X) = D(A)$ is dense, X is WCG. Now complemented subspaces of WCG spaces are trivially WCG again. We conclude that l^∞ is WCG, a contradiction. In fact, every weakly compact set of l^∞ is separable (e.g. note that l^∞ embeds into $L^\infty[0,1]$ and apply [2, Thm. VIII.4.13]). ////

A σ-Dedekind complete Banach lattice not having order continuous norm contains a subspace isomorphic to l^∞ [4, Prop. II.1.a.7]. Hence the following is an immediate consequence of the previous lemma.

Theorem 1.4. *Suppose X is a σ-Dedekind complete Banach lattice. If X is sun-reflexive with respect to a C_0-semigroup $T(t)$, then X has order continuous norm.*

In particular this result applies to Banach function spaces. Finally we will consider *positive* semigroups.

Theorem 1.5. *Suppose $T(t)$ is a positive C_0-semigroup on a Banach function space X. Then $X^\odot \subset X'$ if and only if $f_n \downarrow 0$ implies $\|R(\lambda, A)f_n\| \to 0$.*

Proof: Since $T(t)$ is positive, $R(\lambda, A)$ is positive for λ large enough. Since X' is closed and $X^\odot$ is the closure of $R(\lambda, A)^*(X^*)$, it suffices to prove that for a positive linear operator $T : X \to X$ we have $T^*(X^*) \subset X'$ if and only if $f_n \downarrow 0$ implies $\|Tf_n\| \to 0$. First we prove the 'if'-part. Let $\phi \in X^*$. To prove that $T^*\phi \in X'$, let $f_n \downarrow 0$ in X. By assumption this implies $\|Tf_n\| \to 0$. In particular, $\langle \phi, Tf_n \rangle \to 0$, so $\langle T^*\phi, f_n \rangle \to 0$ and hence $T^*\phi \in X'$. Conversely, assume $T^*X^* \subset X'$. Let $\phi \in X^*$ be positive and suppose $f_n \downarrow 0$ in X. Since $T^*\phi \in X'$ we have $\langle \phi, Tf_n \rangle = \langle T^*\phi, f_n \rangle \to 0$. Since T is positive we actually have $\langle \phi, Tf_n \rangle \downarrow 0$. Since this holds for all positive ϕ, from [9] we deduce $\|Tf_n\| \to 0$. ////

2. The multiplication semigroup

Let $h \in L^0(\mu)$ be a complex-valued measurable function and define the operator A_h by

$$\begin{aligned} D(A_h) &= \{f \in X : hf \in X\}; \\ A_h f &= hf, \quad f \in D(A_h). \end{aligned} \tag{1}$$

Note that A_h is a closed operator. Put

$$E_n = \{s \in \Omega : |h(s)| \leq n\}, \tag{2}$$

let χ_{E_n} be its characteristic function and define the band projections

$$P_n : X \to X, \quad P_n f = \chi_{E_n} f. \tag{3}$$

Since $|P_n f| \leq |f|$ for all f, P_n indeed maps X into X. In fact, from the lattice property of the norm we see immediately that P_n is a contraction mapping.

In general $D(A_h)$ need not be dense, as the example $X = L^\infty(0,1)$, $h(s) = s^{-1}$ shows.

A subset B of $L^0(\mu)$ is called *solid* if the following holds: whenever $|f| \leq |g|$ and $g \in B$ then also $f \in B$. In particular, if B is solid and $f \in B$ then also $|f| \in B$. It is easy to see that the norm-closure if a solid set is solid. An *ideal* is a solid linear subspace. Note that by definition every Banach function space is an ideal in $L^0(\mu)$.

Proposition 2.1. *$D(A_h)$ is solid. Moreover, $D(A_h)$ is dense if and only if $\lim_n \|P_n f - f\| = 0$ for all $f \in X$.*

Proof: Suppose $g \in D(A_h)$ and let $f \in X$ be a function satisfying $|f| \leq |g|$. By assumption $hg \in X$, hence also $|hg| \in X$ since X is an ideal. But $|hf| \leq |hg|$, so $hf \in X$ which implies that $f \in D(A_h)$. This proves the first assertion.

Suppose $\|P_n f - f\| \to 0$ for all $f \in X$. To prove that $D(A_h)$ is dense it suffices to show that $P_n f \in D(A_h)$ for all $f \in X$. But on E_n we have $|h(s)| \leq n$, so

$$|hP_n f| \leq |nP_n f| \leq n|f|$$

showing that $hP_n f \in X$ and hence $P_n f \in D(A_h)$. Conversely, suppose $D(A_h)$ is dense. First let $f \in D(A_h)$. Then

$$|P_n f - f| = |\chi_{(\Omega \setminus E_n)} f| \leq \frac{1}{n}|hf| = \frac{1}{n}|A_h f|.$$

Hence by the lattice property of the norm,

$$\|P_n f - f\| \leq \frac{1}{n}\|A_h f\| \to 0, \quad n \to \infty.$$

Since $D(A_h)$ is dense and $\|P_n\| \leq 1$ for all n, the general case follows from a density argument. ////

Observe that it is an immediate corollary of the above proposition that on the Banach function space $X = L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ equipped with the norm $\|f\| := \max\{\|f\|_{L^1(\mathbb{R})}, \|f\|_{L^\infty(\mathbb{R})}\}$, every multiplication semigroup is uniformly continuous.

We will now characterize those $h \in L^0(\mu)$ which give rise to a generator of a C_0-semigroup.

Theorem 2.2. *A_h generates a C_0-semigroup on $\overline{D(A_h)}$ if and only if $Re\ h \leq K$ for some constant K.*

Proof: Suppose A_h generates a C_0-semigroup $T(t)$ on the closure of $D(A_h)$. Let the sets E_n be defined by (2). If a constant K as above does not exist, then for every n there is a set F_n of positive measure such that $Re\ h > n$ on F_n. Since X is carried by Ω, there are subsets $G_n \subset F_n$ of positive measure such that the characteristic functions χ_{G_n} belong to X. Since $\Omega = \cup_k E_k$, there is a k_n such that $E_{k_n} \cap G_n$ has positive measure. Since

$$\chi_{E_{k_n} \cap G_n} \leq \chi_{G_n}$$

it follows that $\chi_{E_{k_n}\cap G_n} \in X$. Moreover, since $|h| \leq k_n$ on E_{k_n} we have $\chi_{E_{k_n}\cap G_n} \in D(A_h)$, and $\chi_{E_{k_n}\cap G_n}$ is not the zero element of X since $\mu(E_{k_n} \cap G_n) > 0$. Put

$$f_n = \frac{\chi_{E_{k_n}\cap G_n}}{\|\chi_{E_{k_n}\cap G_n}\|}.$$

It is not difficult to see, e.g. from the exponential formula (cf. [1, p.79])

$$T(t)f = \lim_{n\to\infty}\Big(\frac{n}{t}R(\frac{n}{t}, A_h)\Big)^n f, \qquad f \in \overline{D(A_h)},$$

that for almost all s we have

$$T(t)f_n(s) = e^{th(s)}f_n(s).$$

Note that the latter formula makes sense since $f_n \in D(A_h)$ and by assumption $T(t)$ is defined on $\overline{D(A_h)}$. Since $Re\ h > n$ on $E_{k_n} \cap G_n$ we get

$$|T(t)f_n| \geq |e^{nt}f_n|$$

implying

$$\|T(t)\| \geq \|T(t)f_n\| \geq e^{nt}\|f_n\| = e^{nt},$$

a contradiction since this would mean that the operator $T(t)$ is unbounded for each $t > 0$.

Conversely, suppose $Re\ h \leq K$ for some K. Define

$$T(t)f(s) = e^{th(s)}f(s), \quad f \in \overline{D(A_h)}.$$

Then clearly $\|T(t)\| \leq e^{Kt}$. We will show that $T(t)$ is a C_0-semigroup whose generator is A_h. Fix $f \in \overline{D(A_h)}$ and $\epsilon > 0$. Since $D(A_h)$ is solid, so is its closure $\overline{D(A_h)}$; in other words, $\overline{D(A_h)}$ is a Banach function space on its own right. Hence we may apply Proposition 2.1 to obtain an n such that $\|P_n f - f\| < \epsilon$. Now on E_n we have $-n \leq |h| \leq n$. Choose $0 < t_0 \leq 1$ so small that for any $0 \leq t \leq t_0$ and $|\alpha| \leq n$ we have $|e^{\alpha t} - 1| < \epsilon$. Then for such t,

$$\begin{aligned}\|T(t)f - f\| &\leq \|T(t)(f - P_n f)\| + \|f - P_n f\| + \|T(t)P_n f - P_n f\| \\ &\leq (e^{Kt} + 1)\epsilon + \|(e^{ht} - 1)\chi_{E_n} f\| \\ &\leq (e^{|K|} + 1)\epsilon + \epsilon\|\chi_{E_n} f\| \\ &\leq (e^{|K|} + 1 + \|f\|)\epsilon.\end{aligned}$$

Therefore $T(t)$ is strongly continuous on $\overline{D(A_h)}$ and obviously A_h is its generator. ////

We remark that this result could also easily be derived from the Hille-Yosida theorem.

It is an easy consequence of the definition that X has order continuous norm if and only if for all $f \in X$ and decreasing sets $F_1 \supset F_2 \supset \ldots \downarrow \emptyset$ we have $\|f\chi_{F_n}\| \to 0$. Using this equivalent formulation together with Proposition 2.1 and Theorem 2.2 we obtain:

Theorem 2.3. *X has order continuous norm if and only if A_h generates a C_0-semigroup on X for every h whose real part is bounded from above.*

Proof: Suppose X has order continuous norm. Take h with $Re\ h \leq K$ and define the sets E_n and maps P_n according to (2) and (3). Since

$$E_1 \subset E_2 \subset ... \uparrow \Omega,$$

for all $f \in X$ we get

$$\|P_n f - f\| = \|f\chi_{\Omega \setminus E_n}\| \to 0.$$

Hence by Proposition 2.1, $D(A_h)$ is dense. Then Theorem 2.2 shows that A_h is a generator on X.

Conversely, let $\Omega = F_0 \supset F_1 \supset F_2 \supset ... \downarrow \emptyset$. Define $h \in L^0(\mu)$ by

$$h(s) = -n, \quad s \in F_n \setminus F_{n+1}.$$

Then

$$E_n = \{s \in \Omega : |h(s)| \leq n\} = \Omega \setminus F_{n+1}.$$

Since by assumption A_h is a generator on X, hence in particular $D(A_h)$ is dense, we get by Proposition 2.1

$$\|f\chi_{F_{n+1}}\| = \|f\chi_{\Omega\setminus F_{n+1}} - f\| = \|P_n f - f\| \to 0.$$

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From now on we assume h to be fixed with $Re\ h$ bounded from above. If A_h is the generator of a semigroup $T(t)$ on X, then the adjoint $T^*(t)$ is well-defined on X^*. In the following theorem we will give a representation for the semigroup dual $X^\odot$. Let $[P_n^* X^*]_{n=1}^\infty$ denote the closed linear span in X^* of the subspaces $P_n^* X^*$, $n = 1, 2, ...$

Theorem 2.4. $X^\odot = [P_n^* X^*]_{n=1}^\infty$.

Proof: First note that X^* is a Banach lattice, so whenever $\phi \in X^*$, then $|\phi|$ is a well-defined element of X^* of norm $\|\phi\|$. We start by showing that $D(A_h^*)$ is solid. Suppose $|\phi| \leq |\psi|$ with $\psi \in D(A_h^*)$. Clearly,

$$\langle h\phi, f\rangle := \langle \phi, hf\rangle$$

defines a linear functional $h\phi$ on $D(A_h)$ and for $f \in D(A_h)$,

$$\langle h\phi, f\rangle = \langle \phi, hf\rangle \leq \langle |\phi|, |hf|\rangle \leq \langle |\psi|, |hf|\rangle = \langle |h\psi|, |f|\rangle \leq \|A_h^*\psi\|\ \|f\|.$$

Therefore $h\phi$ is bounded on $D(A_h)$. Since $D(A_h)$ is dense, $h\phi$ extends to a bounded linear functional on X. This proves that $\phi \in D(A_h^*)$.

We will now prove the inclusion $[P_n^* X^*]_{n=1}^\infty \subset X^\odot$. Let $\phi \in P_n^* X^*$, say $\phi = P_n^*\psi$. We have to show that $\phi \in X^\odot$. Since $D(A_h^*)$ is solid, so is its closure $X^\odot$. Therefore it suffices to show that $|\phi| \in X^\odot$. Fix $\epsilon > 0$ and choose $t_0 > 0$ so small that for any $0 \leq t \leq t_0$ and $|\alpha| \leq n$ we have $|e^{\alpha t} - 1| < \epsilon$. Since we have $|\phi| = |P_n^*\psi| = P_n^*|\psi|$, and hence for $t \leq t_0$,

$$\begin{aligned} |\langle T^*(t)|\phi| - |\phi|, f\rangle| &= |\langle |\psi|, P_n(e^{th}f - f)\rangle| \\ &= |\langle |\psi|, \chi_{E_n}(e^{th} - 1)f\rangle| \\ &\leq \epsilon\langle |\phi|, |f|\rangle| \\ &\leq \epsilon\|\phi\|\ \|f\|. \end{aligned}$$

Hence

$$\|T^*(t)|\phi| - |\phi|\| \leq \epsilon\|\phi\|$$

showing that $|\phi| \in X^{\odot}$ and therefore also $\phi \in X^{\odot}$. Since $X^{\odot}$ is a closed linear space this implies that $[P_n^* X^*]_{n=1}^{\infty} \subset X^{\odot}$.

To conclude the proof we show the reverse inclusion. Since $\overline{D(A_h^*)} = X^{\odot}$ it suffices to prove that $D(A_h^*) \subset [P_n^* X^*]_{n=1}^{\infty}$. Let $\phi \in D(A_h^*)$. Since $D(A_h^*)$ is solid, we may without loss of generality assume that $\phi \geq 0$. It suffices to prove that $\|P_n^*\phi - \phi\| \to 0$ as $n \to \infty$. For any $f \in D(A_h)$ we have

$$|\langle P_n^*\phi - \phi, f\rangle| = |\langle \phi, \chi_{(\Omega\setminus E_n)} f\rangle| \leq \frac{1}{n}|\langle \phi, |hf|\rangle| = \frac{1}{n}\langle |h\phi|, |f|\rangle \leq \frac{1}{n}\|A^*\phi\|\ \|f\|.$$

This shows that $\|P_n^*\phi - \phi\| \leq n^{-1}\|A_h^*\phi\| \to 0$. ////

Finally we will consider the case where Ω is compact Hausdorff space and μ is a Borel measure. In this case it is natural to see what improvements can be obtained when we require $h \in L^0(\mu)$ to be *continuous*. In fact we will ask something weaker, viz. that $|h|$ is a continuous function $\Omega \to \overline{\mathbb{R}}$, the one-point compactification of $\mathbb{R}$. For such functions we put $E_\infty = \{s \in \Omega : |h(s)| = \infty\}$. Since $h \in L^0(\mu)$, necessarily $\mu(E_\infty) = 0$. We will say that $f \in X$ is *compactly supported* if there is a compact $K \subset \Omega\setminus E_\infty$ such that $f = \chi_K f$ a.e. and we define X_c to be the linear subspace of X consisting of all compactly supported functions. Of course X_c depends on h. A functional $\phi \in X^*$ is said to be compactly supported if there is a compact $K \subset \Omega\setminus E_\infty$ such that $\langle \phi, f\rangle = \langle \phi, \chi_K f\rangle$ for all $f \in X$.

Theorem 2.5. *A_h generates a C_0-semigroup if and only if X_c is dense in X. In this case $X^{\odot}$ is the closure of the compactly supported functionals.*

Proof: Suppose A_h generates a C_0-semigroup. Since $|h|$ is continuous, we see that the sets $E_n \subset \Omega\setminus E_\infty$ defined by (2) are closed in Ω, hence compact. Now take $f \in X$ arbitrary. By assumption $D(A_h)$ is dense, so by Proposition 2.1 we have $\|P_n f - f\| \to 0$. Since $P_n f$ is supported in the compact set E_n, this proves that X_c is dense in X.

For the converse, assume X_c to be dense. In view of Theorem 2.2 we must show that $D(A_h)$ is dense (the convention that $Re\ h \leq K$ is still in force). In fact we will show that $X_c \subset D(A_h)$. Indeed, let $f \in X_c$ be supported in the compact set $K \subset \Omega\setminus E_\infty$. Since $|h|$ is continuous as a function $K \to \mathbb{R}$, we see that h is bounded on K. This implies that $h \in D(A_h)$.

The assertion on $X^{\odot}$ is proved in exactly the same way, using the characterization from Theorem 2.4. ////

Example 2.6. (i) Let $X = L^1(\mathbb{R})$, $h(t) = t$. Letting $\Omega = \overline{\mathbb{R}}$ we conclude from Theorem 2.5 that $X^{\odot}$ is the closed ideal in L^∞ generated by $C_0(\mathbb{R})$.

(ii) Let $X = L^1(D)$ with D the closed unit disc in $\mathbb{C}$. Suppose h is continuous in D with $\lim_{s\to t}|h(s)| = \infty$ for all $t \in \partial D$. Then $X^{\odot}$ is the closed ideal in $L^\infty(D)$ generated by the subpace of continuous functions which are zero on ∂D.

From Theorem 2.4 or 2.5 we immediately deduce the following.

Corollary 2.7. *Let X be a Banach space with an unconditional basis $\{x_n\}_{n=1}^{\infty}$. Then $Ax_n := k_n x_n$ generates a C_0-semigroup if and only if $Re\ k_n \leq K$ for some constant K. If $|k_n| \to \infty$ then $X^{\odot} = [x_n^*]_{n=1}^{\infty}$, the closed linear span of the coordinate functionals.*

Proof: Regard X as a Banach function space on $\Omega = \mathbb{N}$. ////

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